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Strong convergence of an Ishikawa-type algorithm in CAT(0) spaces

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Abstract

We study strong convergence of an Ishikawa-type algorithm of two asymptotically nonexpansive type maps to their common fixed point on a CAT(0) space. Our work provides an affirmative answer to the question of Tan and Xu (Proc. Am. Math. Soc. 122:733-739, 1994); in particular, strong convergence of an Ishikawa-type algorithm of two asymptotically nonexpansive maps without the rate of convergence condition is obtained on a nonlinear domain.

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1 Introduction

A CAT(0) space is simply a geodesic metric space whose each geodesic triangle is at least as thin as its comparison triangle in the Euclidean plane. In 2004, Kirk [1] proved a fixed point theorem for a nonexpansive map defined on a subset of a CAT(0) space. Since then, approximation of fixed points of nonlinear maps on a CAT(0) space has rapidly developed (see, *e.g.*, [2–5]).

We describe briefly the needed details for a CAT(0) space. A metric space (X,d) is said to be a *length space* if any two points of X are joined by a rectifiable path (that is, a path of finite length) and the distance between any two points of X is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, d is said to be a *length metric* (otherwise known as an *inner metric* or *intrinsic metric*). In case no rectifiable path joins two points of the space, the distance between them is taken to be ∞.

A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say that X is: (i) a *geodesic space* if any two points of X are joined by a geodesic, and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x,y]$, called the segment joining x to y.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists (see [6]).

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x,y) \leq d(\bar{x},\bar{y}).$$

If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \leq \frac{1}{2} d(x,y_1)^2 + \frac{1}{2} d(x,y_2)^2 - \frac{1}{4} d(y_1,y_2)^2.$$

The above inequality is the (CN) inequality of Bruhat and Titz [7] and it was extended in [8] as follows:

$$d\big(z, \alpha x \oplus (1-\alpha)y\big)^2 \leq \alpha d(z,x)^2 + (1-\alpha) d(z,y)^2 - \alpha(1-\alpha) d(x,y)^2$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$.

Let us recall that *a geodesic metric space is a* CAT(0) *space if and only if it satisfies the* (*CN*) *inequality* (*see* [6], *p.*163). Moreover, if X is a CAT(0) metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d\big(z, \alpha x \oplus (1-\alpha)y\big) \leq \alpha d(z,x) + (1-\alpha) d(z,y)$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset C of a CAT(0) space X is convex if for any $x, y \in C$, we have $[x,y] \subset C$.

Complete CAT(0) spaces are known as *Hadamard spaces* (*see* [9]). The reader interested in a more general nonlinear domain, namely 2-uniformly convex hyperbolic space containing a CAT(0) space as a special case, is referred to Dehaish [10] and Dehaish *et al.* [11].

Let C be a nonempty subset of a metric space (X,d). Then a selfmap T on C is:

(i) uniformly L-Lipschitzian if for some $L > 0$, $d(T^n x, T^n y) \leq L d(x,y)$ for $x, y \in C$, $n \geq 1$;

(ii) uniformly Hölder continuous if for some positive constants L and α, $d(T^n x, T^n y) \leq L d(x,y)^\alpha$ for $x, y \in C$, $n \geq 1$;

(iii) uniformly equicontinuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(T^n x, T^n y) \leq \varepsilon$ whenever $d(x,y) \leq \delta$ for $x, y \in C$, $n \geq 1$ or, equivalently, T is uniformly equicontinuous if and only if $d(T^n x_n, T^n y_n) \to 0$ whenever $d(x_n, y_n) \to 0$ as $n \to \infty$;

(iv) asymptotically nonexpansive if there is a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n d(x,y)$ for $x, y \in C$, $n \geq 1$;

(v) asymptotically nonexpansive in the intermediate sense provided T is uniformly continuous and $\limsup_{n\to\infty} \sup_{x,y\in C} \{d(T^n x, T^n y) - d(x,y)\} \leq 0$ for $n \geq 1$, and

(vi) of asymptotically nonexpansive type in the sense of Xu [12] if
$\limsup_{n\to\infty} \sup_{x\in C}\{d(T^n x, T^n y) - d(x,y)\} \leq 0$ for each $y \in C$, $n \geq 1$;

(vii) of asymptotically nonexpansive type in the sense of Chang *et al.* [13] if
$\limsup_{n\to\infty} \sup_{x\in C}\{d(T^n x, T^n y)^2 - d(x,y)^2\} \leq 0$ for each $y \in C$, $n \geq 1$.

The map T is semi-compact if for any bounded sequence $\{x_n\}$ in C with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in C$ as $n_i \to \infty$.

It is not difficult to see that nonexpansive map, asymptotically nonexpansive map, asymptotically nonexpansive map in the intermediate sense and asymptotically nonexpansive type map in the sense of Xu [12] all are special cases of asymptotically nonexpansive type map in the sense of Chang *et al.* [13]. Moreover, a uniformly L-Lipschitzian map is uniformly Hölder continuous, and a uniformly Hölder continuous map is uniformly equicontinuous. However, the converse statements are not true as indicated below.

Example 1.1 Take $X = \mathbb{R}$ and $C = [0,1]$. Define $T : C \to C$ by $Tx = (1 - x^{\frac{3}{2}})^{\frac{2}{3}}$ for all $x \in C$. Then T is uniformly equicontinuous, but it is neither uniformly L-Lipschitzian nor uniformly Hölder continuous.

In uniformly convex Banach spaces, the convergence of an Ishikawa-type algorithm and a Mann-type algorithm of nonexpansive maps, asymptotically nonexpansive maps and asymptotically nonexpansive maps in the intermediate sense to their fixed points have been studied by a number of researchers [12, 14–24]. For the iterative construction of fixed points of some other classes of nonlinear maps, see [25–27].

The sequence $\{k_n\}$ in definition (iv) satisfies the rate of convergence condition if $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. This condition has been extensively used in iterative construction of fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces and CAT(0) spaces (see, *e.g.*, [4, 5, 21, 28, 29]).

Chang *et al.* [13] established strong convergence of an Ishikawa-type algorithm as well as a Mann-type algorithm to a fixed point of an asymptotically nonexpansive type map.

We shall follow the idea of a geodesic path, namely, there exists a unique point $\alpha x \oplus (1-\alpha)y$ for any $x, y \in C$ and $\alpha \in [0,1]$, to construct an Ishikawa-type algorithm of two asymptotically nonexpansive type maps on a nonempty subset C of a CAT(0) space.

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n S^n y_n, \\ &y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $0 \leq \alpha_n, \beta_n \leq 1$.

When $T = I$ (the identity map) in (1.1), it reduces to the following Mann-type algorithm:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $0 \leq \alpha_n \leq 1$.

The purpose of this paper is to approximate a common fixed point of asymptotically nonexpansive type maps in a special kind of a metric space, namely a CAT(0) space. Our work is a significant generalization of the corresponding results in [5], and it provides analogues of the related results of Chang *et al.* [13] in uniformly convex Banach spaces. One of our results (Theorem 2.4) gives an affirmative answer to a famous question of Tan and Xu [30] on a nonlinear domain for common fixed points.

2 Fixed point approximation

We begin with the following asymptotic regularity result.

Lemma 2.1 *Let C be a nonempty bounded closed convex subset of a* CAT(0) *space X. Let $S, T : C \to C$ be uniformly equicontinuous. Then for the sequence $\{x_n\}$ in* (1.1) *satisfying*

$$\lim_{n\to\infty} d\bigl(x_n, S^n x_n\bigr) = 0 = \lim_{n\to\infty} d\bigl(x_n, T^n x_n\bigr),$$

we have that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n).$$

Proof Since S is uniformly equicontinuous and

$$\begin{aligned} d(x_n, y_n) &= d\bigl(x_n, (1-\beta_n)x_n \oplus \beta_n T^n x_n\bigr) \\ &\le (1-\beta_n)d(x_n, x_n) + \beta_n d\bigl(x_n, T^n x_n\bigr) \\ &= \beta_n d\bigl(x_n, T^n x_n\bigr) \to 0, \end{aligned}$$

therefore,

$$d\bigl(S^n x_n, S^n y_n\bigr) \to 0.$$

Now

$$\begin{aligned} d(x_n, x_{n+1}) &= d\bigl(x_n, (1-\alpha_n)x_n \oplus \alpha_n S^n y_n\bigr) \\ &\le \alpha_n d\bigl(x_n, S^n y_n\bigr) \\ &\le d\bigl(x_n, S^n x_n\bigr) + d\bigl(S^n x_n, S^n y_n\bigr) \end{aligned}$$

gives that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.1}$$

Clearly,

$$\begin{aligned} d(x_n, Sx_n) \le{}& d(x_n, x_{n+1}) + d\bigl(x_{n+1}, S^{n+1}x_{n+1}\bigr) \\ &+ d\bigl(S^{n+1}x_{n+1}, S^{n+1}x_n\bigr) + d\bigl(S^{n+1}x_n, Sx_n\bigr), \end{aligned} \tag{2.2}$$

applying lim sup to both sides of (2.2), using the uniformly equicontinuous property of S and (2.1), we get that

$$\limsup_{n\to\infty} d(x_n, Sx_n) \le 0$$

and hence

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

Similarly,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is,

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n). \qquad \square$$

Our main result is as follows.

Theorem 2.2 *Let C be a nonempty, bounded, closed and convex subset of a* CAT(0) *space X. Let $S, T : C \to C$ be uniformly equicontinuous and asymptotically nonexpansive type maps such that $F(S) \cap F(T) \neq \emptyset$. Suppose that $0 < \delta \leq \alpha_n, \beta_n \leq 1 - \delta$ for some $\delta \in (0,1)$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the control parameters of the iteration scheme $\{x_n\}$ in* (1.1). *If S or T is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof For any $p \in F(S) \cap F(T)$, by the (CN)-inequality, we have

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d\bigl(\alpha_n x_n \oplus \alpha_n S^n y_n, p\bigr)^2 \\
&\leq (1-\alpha_n) d(x_n, p)^2 + \alpha_n d\bigl(S^n y_n, p\bigr)^2 \\
&\quad - \alpha_n(1-\alpha_n) d\bigl(x_n, S^n y_n\bigr)^2 \\
&= d(x_n, p)^2 + \alpha_n \bigl\{ d\bigl(S^n y_n, p\bigr)^2 - d(y_n, p)^2 \bigr\} \\
&\quad + \alpha_n \bigl\{ d(y_n, p)^2 - d(x_n, p)^2 \bigr\} \\
&\quad - \alpha_n(1-\alpha_n) d\bigl(x_n, S^n y_n\bigr)^2.
\end{aligned}$$

That is,

$$\begin{aligned}
d(x_{n+1}, p)^2 &\leq d(x_n, p)^2 + \alpha_n \bigl\{ d\bigl(S^n y_n, p\bigr)^2 - d(y_n, p)^2 \bigr\} \\
&\quad + \alpha_n \bigl\{ d(y_n, p)^2 - d(x_n, p)^2 \bigr\} \\
&\quad - \alpha_n(1-\alpha_n) d\bigl(x_n, S^n y_n\bigr)^2. \qquad (2.3)
\end{aligned}$$

Next we consider the third term on the right side of (2.3):

$$\begin{aligned}
d(y_n, p)^2 - d(x_n, p)^2 &= d\bigl((1-\beta_n)x_n \oplus \beta_n T^n x_n, p\bigr)^2 - d(x_n, p)^2 \\
&\leq (1-\beta_n) d(x_n, p)^2 + \beta_n d\bigl(T^n x_n, p\bigr)^2 - d(x_n, p)^2 \\
&\quad - \beta_n(1-\beta_n) d\bigl(x_n, T^n x_n\bigr)^2 \\
&= \beta_n \bigl\{ d\bigl(T^n x_n, p\bigr)^2 - d(x_n, p)^2 \bigr\} \\
&\quad - \beta_n(1-\beta_n) d\bigl(x_n, T^n x_n\bigr)^2.
\end{aligned}$$

That is,

$$\begin{aligned}\alpha_n\{d(y_n,p)^2 - d(x_n,p)^2\} &\le \alpha_n\beta_n\{d(T^nx_n,p)^2 - d(x_n,p)^2\}\\ &\quad - \alpha_n\beta_n(1-\beta_n)d(x_n,T^nx_n)^2. \qquad (2.4)\end{aligned}$$

Substituting (2.4) into (2.3) and using $0 < \delta \le \alpha_n, \beta_n \le 1-\delta$, we have

$$\begin{aligned}&d(x_{n+1},p)^2\\ &\quad\le d(x_n,p)^2 - \frac{\alpha_n(1-\alpha_n)}{2}d(S^ny_n,p)^2\\ &\qquad - \frac{\alpha_n\beta_n(1-\beta_n)}{2}d(x_n,T^nx_n)^2\\ &\qquad + \alpha_n\left\{d(S^ny_n,p)^2 - d(y_n,p)^2 - \frac{(1-\alpha_n)}{2}d(S^ny_n,p)^2\right\}\\ &\qquad + \alpha_n\beta_n\left\{d(T^nx_n,p)^2 - d(x_n,p)^2 - \frac{(1-\beta_n)}{2}d(x_n,T^nx_n)^2\right\}\\ &\quad\le d(x_n,p)^2 - \frac{\delta^2}{2}d(S^ny_n,p)^2 - \frac{\delta^3}{2}d(x_n,T^nx_n)^2\\ &\qquad + (1-\delta)\left\{d(S^ny_n,p)^2 - d(y_n,p)^2 - \frac{\delta}{2}d(S^ny_n,p)^2\right\}\\ &\qquad + (1-\delta)^2\left\{d(T^nx_n,p)^2 - d(x_n,p)^2 - \frac{\delta}{2}d(x_n,T^nx_n)^2\right\}. \qquad (2.5)\end{aligned}$$

Next we prove that

$$\lim_{n\to\infty} d(x_n,S^ny_n) = 0 = \lim_{n\to\infty} d(x_n,T^nx_n).$$

Assume that $\limsup_{n\to\infty} d(x_n,S^ny_n) > 0$ and $\limsup_{n\to\infty} d(x_n,T^nx_n) > 0$.

Then there exist subsequences (we use the same notation for a subsequence as well) of $\{x_n\}$, $\{y_n\}$ and $\mu_1 > 0$, $\mu_2 > 0$ such that $d(x_n,S^ny_n) \ge \mu_1 > 0$ and $d(x_n,T^nx_n) \ge \mu_2 > 0$.

Now from (2.5) it follows that

$$\begin{aligned}d(x_{n+1},p)^2 &\le d(x_n,p)^2 - \frac{\delta^2\mu_1^2}{2} - \frac{\delta^3\mu_2^2}{2}\\ &\quad + (1-\delta)\left\{d(S^ny_n,p)^2 - d(y_n,p)^2 - \frac{\delta\mu_1^2}{2}\right\}\\ &\quad + (1-\delta)^2\left\{d(T^nx_n,p)^2 - d(x_n,p)^2 - \frac{\delta\mu_2^2}{2}\right\}. \qquad (2.6)\end{aligned}$$

For an asymptotically nonexpansive type map T, we have that

$$\limsup_{n\to\infty}\sup_{x\in C}\{d(T^nx,p)^2 - d(x,p)^2\} \le 0.$$

That is,

$$\lim_{n\to\infty}\sup_{m\ge n}\left\{\sup_{x\in C}\left(d(T^mx,p)^2 - d(x,p)^2\right)\right\} \le 0.$$

Hence, for given $\frac{\delta\mu_i^2}{2} > 0$ ($i = 1, 2$), there exists a positive integer n_0 such that

$$\sup_{n\geq n_0}\left\{\sup_{x\in C}\left(d\left(T^n x, p\right)^2 - d(x,p)^2\right)\right\} < \frac{\delta\mu_i^2}{2}.$$

Since $\{x_n\}$ and $\{y_n\}$ are sequences in C, therefore, for $n \geq n_0$, it follows that

$$d\left(S^n y_n, p\right)^2 - d(y_n, p)^2 < \frac{\delta\mu_1^2}{2}$$

and

$$d\left(T^n x_n, p\right)^2 - d(x_n, p)^2 < \frac{\delta\mu_2^2}{2}.$$

In the light of the two inequalities above, (2.6) reduces to

$$\frac{\delta^2\mu_1^2}{2} + \frac{\delta^3\mu_2^2}{2} \leq d(x_n, p)^2 - d(x_{n+1}, p)^2 \quad \text{for all } n \geq n_0. \tag{2.7}$$

Let $m \geq n_0$ be any positive integer. Obtain $m - n_0$ inequalities from (2.7) and then, summing up these inequalities, we get

$$\begin{aligned}\left(\frac{\delta^2\mu_1^2}{2} + \frac{\delta^3\mu_2^2}{2}\right)(m - n_0) &\leq d(x_{n_0}, p)^2 - d(x_{m+1}, p)^2 \\ &\leq d(x_{n_0}, p)^2 < \infty.\end{aligned}$$

If $m \to \infty$, then

$$\infty = d(x_{n_0}, p)^2 < \infty,$$

a contradiction.

This proves that $\limsup_{n\to\infty} d(x_n, S^n y_n) = 0 = \limsup_{n\to\infty} d(x_n, T^n x_n)$.

That is,

$$\lim_{n\to\infty} d\left(x_n, S^n y_n\right) = 0 = \lim_{n\to\infty} d\left(x_n, T^n x_n\right).$$

As

$$d\left(x_n, S^n x_n\right) \leq d\left(x_n, S^n y_n\right) + d\left(S^n x_n, S^n y_n\right),$$

$d(x_n, y_n) \to 0$ and S is uniformly equicontinuous. So, by taking lim sup on both sides, we get

$$\lim_{n\to\infty} d\left(x_n, S^n x_n\right) = 0.$$

Now, Lemma 2.1 implies that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n). \tag{2.8}$$

Since T is semi-compact, therefore there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $q \in C$ such that

$$x_{n_i} \to q. \tag{2.9}$$

Now, by the uniform equicontinuity of S and T and hence continuity, it follows from (2.8) that

$$d(q, Sq) = 0 = d(q, Tq).$$

This gives that q is a common fixed point of S and T.

We now proceed to establish strong convergence of $\{x_n\}$ to q.

Since

$$d(T^{n_i}x_{n_i}, q) \le d(T^{n_i}x_{n_i}, x_{n_i}) + d(x_{n_i}, q),$$

therefore

$$T^{n_i}x_{n_i} \to q \quad \text{as } n_i \to \infty. \tag{2.10}$$

Clearly,

$$\begin{aligned} d(y_{n_i}, q) &= d\big((1-\beta_{n_i})x_{n_i} \oplus \beta_{n_i}T^{n_i}x_{n_i}, q\big) \\ &\le (1-\beta_{n_i})d(x_{n_i}, q) + \beta_{n_i}d\big(T^{n_i}x_{n_i}, q\big). \end{aligned}$$

Therefore, from (2.9) and (2.10), it follows that

$$y_{n_i} \to q \quad \text{as } n_i \to \infty.$$

Next we prove that $S^{n_i}y_{n_i} \to q$ as $n_i \to \infty$.

Since $S : C \to C$ is of asymptotically nonexpansive type and $\{y_{n_i}\}$ is a sequence in C, therefore we have

$$\begin{aligned} &\limsup_{n_i\to\infty}\big\{d\big(S^{n_i}y_{n_i}, q\big)^2 - d(y_{n_i}, q)^2\big\} \\ &\quad\le \limsup_{n_i\to\infty}\sup_{x\in C}\big\{d\big(S^{n_i}x, q\big)^2 - d(x, q)^2\big\} \\ &\quad\le \limsup_{n\to\infty}\sup_{x\in C}\big\{d\big(S^{n}x, q\big)^2 - d(x, q)^2\big\} \\ &\quad\le 0. \end{aligned} \tag{2.11}$$

As $y_{n_i} \to q$ as $n_i \to \infty$, it follows from (2.11) that

$$\limsup_{n_i\to\infty} d\big(S^{n_i}y_{n_i}, q\big)^2 \le 0.$$

That is,

$$S^{n_i}y_{n_i} \to q \quad \text{as } n_i \to \infty.$$

Replace p by q in (2.5) to get

$$\begin{aligned} d(x_{n_i+1},q)^2 \leq{}& d(x_{n_i},q)^2 - \frac{\delta^2}{2} d\big(S^{n_i}y_{n_i},q\big)^2 - \frac{\delta^3}{2} d\big(x_{n_i},T^{n_i}x_{n_i}\big)^2 \\ &+ (1-\delta)\left\{ d\big(S^{n_i}y_{n_i},q\big)^2 - d(y_{n_i},q)^2 - \frac{\delta}{2} d\big(S^{n_i}y_{n_i},q\big)^2 \right\} \\ &+ (1-\delta)^2\left\{ d\big(T^{n_i}x_{n_i},q\big)^2 - d(x_{n_i},q)^2 - \frac{\delta}{2} d\big(x_{n_i},T^{n_i}x_{n_i}\big)^2 \right\}, \end{aligned}$$

which gives that $x_{n_i+1} \to q$ as $n_i \to \infty$.

Continuing in this way, by induction, we can prove that for any $m \geq 0$,

$$x_{n_i+m} \to q \quad \text{as } n_i \to \infty.$$

By induction, one can prove that $\bigcup_{m=0}^{\infty}\{x_{n_i+m}\}$ converges to q as $i \to \infty$; in fact $\{x_n\}_{n=n_1}^{\infty} = \bigcup_{m=0}^{\infty}\{x_{n_i+m}\}_{i=1}^{\infty}$ gives that $x_n \to q$ as $n \to \infty$. □

We need the following lemma to approximate a common fixed point of two asymptotically nonexpansive maps.

Lemma 2.3 *Every asymptotically nonexpansive selfmap T on a nonempty bounded subset C of a metric space X is uniformly equicontinuous and of asymptotically nonexpansive type.*

Proof Let $T : C \to C$ be an asymptotically nonexpansive map with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $\varepsilon > 0$. Then, for each $\gamma > 0$, there exists a positive integer n_0 such that $k_n - 1 < \gamma$ for all $n \geq n_0$. Put $s = \max\{1+\gamma, k_1, k_2, \ldots, k_{n_0}\}$. Then $d(T^n x, T^n y) \leq k_n d(x,y) \leq sd(x,y)$ for $x,y \in C$, $n \geq 1$. Choose $\delta = \frac{\varepsilon}{s}$. Then $d(T^n x, T^n y) \leq \varepsilon$ whenever $d(x,y) \leq \delta$ for $x,y \in C$, $n \geq 1$, proving that T is uniformly equicontinuous.

The second part of the lemma follows from

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{x\in C} \big\{ d^2\big(T^n x, T^n y\big) - d^2(x,y) \big\} \\ &\quad\leq \lim_{n\to\infty} (k_n - 1) \sup_{x\in C} d^2(x,y) \\ &\quad= 0. \sup_{x\in C} d^2(x,y) \\ &\quad= 0. \end{aligned}$$

□

By Theorem 2.2 and Lemma 2.3, we have the following result which is new in the literature and sets an analogue of Theorem 2 in [21] without the rate of convergence condition.

Theorem 2.4 *Let C be a nonempty, bounded, closed and convex subset of a* CAT(0) *space X. Let $S,T : C \to C$ be asymptotically nonexpansive maps with sequences $\{s_n\}, \{t_n\} \subseteq [1,\infty)$, respectively and $F(S) \cap F(T) \neq \emptyset$. Suppose that $0 < \delta \leq \alpha_n, \beta_n \leq 1-\delta$ for some $\delta \in (0,1)$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the control parameters of the sequence $\{x_n\}$ in* (1.1). *If S or T is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

As every uniformly equicontinuous map is uniformly L-Lipschitzian, so the following result is immediate and it unifies Theorem 2.1 and Theorem 2.2 of Chang *et al.* [13] in Hadamard spaces.

Theorem 2.5 *Let C be a nonempty, bounded, closed and convex subset of a* CAT(0) *space X. Let $S, T : C \to C$ be uniformly L-Lipschitzian and asymptotically nonexpansive type maps such that $F(S) \cap F(T) \neq \emptyset$. Suppose that* $0 < \delta \leq \alpha_n, \beta_n \leq 1 - \delta$ *for some* $\delta \in (0,1)$, *where $\{\alpha_n\}$ and $\{\beta_n\}$ are the control parameters of the sequence $\{x_n\}$ in* (1.1). *If S or T is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

For $S = T$, Theorem 2.5 sets an analogue of Theorem 2.1 in [13].

Theorem 2.6 *Let C be a nonempty, bounded, closed and convex subset of a* CAT(0) *space X. Let $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically nonexpansive type map such that $F(T) \neq \emptyset$. Suppose that* $0 < \delta \leq \alpha_n, \beta_n \leq 1 - \delta$ *for some* $\delta \in (0,1)$, *where $\{\alpha_n\}$ and $\{\beta_n\}$ are the control parameters of the sequence $\{x_n\}$ in* (1.1) *with $S = T$. If T is semi-compact, then $\{x_n\}$ converges strongly to a fixed point of T.*

On taking $S = I$ (the identity map) in Theorem 2.5, we obtain an analogue of Theorem 2.2 in [13].

Theorem 2.7 *Let C be a nonempty, bounded, closed and convex subset of a* CAT(0) *space X. Let $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically nonexpansive type map such that $F(T) \neq \emptyset$. Suppose that* $0 < \delta \leq \alpha_n, \beta_n \leq 1 - \delta$ *for some* $\delta \in (0,1)$, *where $\{\alpha_n\}$ and $\{\beta_n\}$ are the control parameters of the sequence $\{x_n\}$ in* (1.2). *If T is semi-compact, then $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 2.8 (1) Tan and Xu [30] obtained only weak convergence theorems for asymptotically nonexpansive maps satisfying the rate of convergence condition and remarked, 'We do not know whether our weak convergence Theorem 3.1 remains valid if k_n is allowed to approach 1 slowly enough so that $\sum_{n=1}^{\infty}(k_n - 1)$ diverges'. Our Theorem 2.4 gives an affirmative answer to their question in CAT(0) spaces.

(2) Our results are generalizations in CAT(0) spaces of the corresponding basic results in [16, 21, 28, 29].

(3) Theorem 2.2 improves and generalizes Theorems 4.2-4.3 in [5].

Competing interests

The author did not provide this information.

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